Consider the following ODE: $z\frac{dN_\phi}{dz} + N_\phi - Az = 0$ where $A = \sec\beta(p_O \tan\beta - p_\phi) = \text{constant}$

This will be solved using the 'exact solution' method (*James*, 2001).

The above ODE may be expressed in the following form: $p(z, N_\phi)\frac{dN_\phi}{dz} + q(z, N_\phi) = 0$

Let $h(z, N_\phi)$ be the exact solution such that $\frac{\partial h}{\partial N_\phi} = p(z, N_\phi)$ and $\frac{\partial h}{\partial z} = q(z, N_\phi)$

Thus: $p(z, N_\phi) = z \quad \therefore \quad \frac{\partial h}{\partial N_\phi} = z \quad \Rightarrow \quad h = z\int_0^{N_\phi} \partial N_\phi = zN_\phi$

and: $q(z, N_\phi) = N_\phi - Az \quad \therefore \quad \frac{\partial h}{\partial z} = N_\phi - Az \quad \Rightarrow \quad h = \int_0^z (N_\phi - Az)\partial z = zN_\phi - \frac{1}{2}z^2 A$

Combining the two: $h(z, N_\phi) = zN_\phi - \frac{1}{2}z^2 A$

Hence the original ODE becomes: $\frac{d}{dz}(zN_\phi - \frac{1}{2}z^2 A) = 0$

and it's general solution is: $zN_\phi - \frac{1}{2}z^2 A = C$ where $C$ is a constant

Boundary condition: $N_\phi = 0$ when $z = 0 \quad \therefore \quad C = 0$

hence: $zN_\phi - \frac{1}{2}z^2 A = 0$

or: $N_\phi(z) = \frac{1}{2}zA = \frac{1}{2}z\sec\beta(p_O \tan\beta - p_\phi)$ Q.E.D