Firstly, some rearranging:

$$z\frac{dN_\phi}{dz} + N_\phi = z\sec\beta\tan\beta\, p_n - z\sec\beta\, p_\phi \quad \text{where} \quad p_n = p_1 + (p_2 - p_1)\frac{(L-z)}{L} = p_2 + \frac{z}{L}(p_1 - p_2)$$

Let $A = \sec\beta\tan\beta$ and $B = \sec\beta$, where A and B are constants.

Hence: $z\frac{dN_\phi}{dz} + N_\phi = zAp_n - zBp_\phi = zA(p_2 + \frac{z}{L}(p_1 - p_2)) - zBp_\phi$

$$= z(Ap_2 - Bp_\phi) + z^2\frac{A}{L}(p_1 - p_2)$$

Let $C = (Ap_2 - Bp_\phi)$ and $D = \frac{A}{L}(p_1 - p_2)$, where C and D are constants.

Hence: $z\frac{dN_\phi}{dz} + N_\phi = Cz + Dz^2$

After all that, consider the following ODE: $z\frac{dN_\phi}{dz} + N_\phi - Cz - Dz^2 = 0$

Using the same 'exact solution' method ($James$, 2001), the above ODE may be expressed in the following form: $p(z, N_\phi)\frac{dN_\phi}{dz} + q(z, N_\phi) = 0$

Let $h(z, N_\phi)$ be the exact solution such that $\frac{\partial h}{\partial N_\phi} = p(z, N_\phi)$ and $\frac{\partial h}{\partial z} = q(z, N_\phi)$

Thus: $p(z, N_\phi) = z \quad \therefore \quad \frac{\partial h}{\partial N_\phi} = z \quad \Rightarrow \quad h = z\int_0^{N_\phi} \partial N_\phi = zN_\phi$

and: $q(z, N_\phi) = N_\phi - Cz - Dz^2 \quad \therefore \quad \frac{\partial h}{\partial z} = N_\phi - Cz - Dz^2$

$$\Rightarrow \quad h = \int_0^z (N_\phi - Cz - Dz^2)\partial z = zN_\phi - \frac{1}{2}z^2C - \frac{1}{3}z^3D$$

Combining the two: $h(z, N_\phi) = zN_\phi - \frac{1}{2}z^2C - \frac{1}{3}z^3D$

Hence the original ODE becomes: $\frac{d}{dz}(zN_\phi - \frac{1}{2}z^2C - \frac{1}{3}z^3D) = 0$

and it's general solution is: $zN_\phi - \frac{1}{2}z^2C - \frac{1}{3}z^3D = K$ where K is a constant

Boundary condition: $N_\phi = 0$ when $z = 0 \quad \therefore \quad K = 0$

hence: $zN_\phi - \frac{1}{2}z^2C - \frac{1}{3}z^3D = 0$

Thus: $N_\phi(z) = \frac{1}{2}zC + \frac{1}{3}z^2D = \frac{1}{2}z\sec\beta(\tan\beta\, p_2 - p_\phi) + \frac{1}{3}z^2(\frac{1}{L}\sec\beta\tan\beta(p_1 - p_2))$

or: $N_\phi(z) = \dfrac{z[\tan\beta(p_2 + \frac{2}{3L}z(p_1 - p_2)) - p_\phi]}{2\cos\beta}$ Q.E.D